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Graded contractions of Casimir operators

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Abstract. We describe graded contractions of Casimir operators of Lie algebras. The formalism applies to any Casimir operator given as a symmetric form. We deal with the quadratic Casimir operator in exhaustive detail and indicate the modifications for operators of higher degree.

1. Introduction

Contractions of Lie algebras are of interest in physics since they provide a natural way for passing from one group of symmetries to another similar, but not otherwise directly related, group of symmetries. The traditional approach to this subject goes under the name of Wigner–Inönü [1] contractions. Recently, a very different approach [2, 3] has been developed involving the concept of grading—it includes the Wigner–Inönü contractions and many more. In addition to graded contractions of algebras, graded contractions of representations and of tensor products have also been obtained.

In this paper we address the question of graded contractions of Casimir operators, which involves grading the universal enveloping algebra. We consider Casimir operators $C^{(k)}$ given as symmetric and homogeneous of degree k polynomials in the generators. Whether the formalism can be extended to generalized Casimir operators, given by more complicated functions of the generators, is an open question.

The Lie algebras L that we consider here are over the real or complex number field and the grading group G is any cyclic group. We wish to stress the extreme generality of the formalism: we treat simultaneously all Lie algebras that admit a chosen grading. In particular, one needs neither to fix the dimension of the algebra (finite or infinite) nor to make a distinction between a Lie algebra and a Lie superalgebra.

In section 2 we give a brief summary of graded contractions of Lie algebras and in section 3 deal with the grading of the universal enveloping algebra and the Casimir operators. Our main results are given in section 4, where we describe graded contractions of the quadratic Casimir operator. Section 5 is devoted to contractions of Casimir operators of higher degree.

2. Graded contractions of Lie algebras

Graded contractions are defined as contractions which preserve a chosen grading.

A grading of a Lie algebra L by a cyclic group G of order N implies the following:

(1) The Lie algebra is decomposed as a linear space into a direct sum of (grading) subspaces

$$L = \oplus L_j \tag{2.1}$$

which are eigenspaces of the action of the generating element g of G on L,

$$L_j = \{x | x \in L,\ gxg^{-1} = e^{2\pi i j/N} x\} . \tag{2.2}$$

(2) The commutation relations in L have a graded structure, i.e. for every choice of elements $x \in L_j$ and $y \in L_k$ we have

$$[x, y] = z \tag{2.3}$$

where z belongs to the grading subspace L_{j+k} as long as the commutator differs from zero. We write for (2.3) symbolically

$$0 \neq [L_j, L_k] \subseteq L_{j+k} . \tag{2.4}$$

Throughout this paper all grading labels are to be read mod N, *where N is the order of the cyclic group G.*

The contraction L^ϵ of L is defined by modification of the commutators of L. We define

$$[L_j, L_k]_\epsilon \equiv \epsilon_{jk} [L_j, L_k] \subseteq \epsilon_{jk} L_{j+k} . \tag{2.5}$$

Thus, the contracted commutator is given by the uncontracted commutator multiplied explicitly by the contraction parameter ϵ_{jk}. It follows from this definition that

$$\epsilon_{jk} = \epsilon_{kj} . \tag{2.6}$$

The familiar Wigner–Inönü contractions are specializations of (2.5) in two ways at once: the grading group is Z_2 and for ϵ_{jk} one uses the ansatz

$$\epsilon_{jk} = a_j a_k / a_{j+k} . \tag{2.7}$$

The parameters ϵ_{jk} are not all free as the contracted commutators must satisfy the Jacobi identity. This leads to the requirement

$$\epsilon_{km} \epsilon_{j,k+m} = \epsilon_{mj} \epsilon_{k,m+j} = \epsilon_{jk} \epsilon_{m,j+k} . \tag{2.8}$$

The solutions of the above equations provide the contractions L^ϵ of any Lie algebra L graded by G. One class of solutions is given by the Wigner–Inönü ansatz in (2.7). Equations (2.8) apply to the generic case. In the special case when some of the commutators vanish identically (i.e. for every choice of elements from the grading subspaces) the corresponding equalities have to be *omitted* from (2.8).

3. The universal enveloping algebra

By definition, Casimir operators are polynomials in the generators of L, which commute with all the generators. They are therefore in the universal enveloping algebra $U(L)$ consisting of 1 and multinomials in the generators. The grading of L, (2.2), provides a G-grading of $U(L)$:

$$U = \oplus U_j . \tag{3.1}$$

As is clear from the definition, any linear combination of Casimir operators is again a Casimir operator so that some convention for defining a basis is needed. Let C denote a particular member from a basis of Casimir operators. Our first result is that C belongs to a single grading subspace of U,

$$C \in U_\ell \qquad \text{for some } \ell \,. \tag{3.2}$$

To prove this we suppose that instead we have, say,

$$C = A + B \qquad A \in U_\ell \qquad B \in U_k \qquad \ell \neq k \,. \tag{3.3}$$

Since C is a Casimir operator it commutes with all $z \in L_m$ for all m:

$$[z, A] + [z, B] = 0 \,. \tag{3.4}$$

But

$$[z, A] \in U_{m+\ell} \qquad [z, B] \in U_{m+k} \tag{3.5}$$

so that the two terms in (3.4) cannot cancel each other but must each vanish. That means that A alone and B alone is a Casimir operator, in contradiction with the assumption that C was a member of a basis.

The formalism we are about to describe is applicable for the grading label ℓ in (3.2) having any allowed value. The details, however, simplify for $\ell = 0$. We observe that ℓ must be zero if the grading group G arises from an *inner* automorphism of the Lie algebra L. In that case the grading element $g \in G$ is some function of the generators of L and therefore commutes with the Casimir operator C. If the grading group G arises from an *outer* automorphism, ℓ need not be zero.

4. Contraction of the quadratic Casimir operator

We now describe the formalism in detail on the example of the quadratic Casimir operator $C^{(2)}$. To be specific, we suppose that the grading group G is Z_N and that

$$C^{(2)} \in U_0 \,. \tag{4.1}$$

Quite explicitly we have

$$C^{(2)} = \sum_{k=0}^{p} P_{k,-k} \qquad p = [N/2] \tag{4.2}$$

where $[\,\cdot\,]$ denotes the greatest integer function and

$$P_{k,-k} \equiv \sum_i \left(x^i y^i + y^i x^i\right) \qquad x^i \in L_k \qquad y^i \in L_{-k} \,. \tag{4.3}$$

It is crucial to our argument that $C^{(2)}$ is a *symmetric homogeneous* of degree two polynomial in the generators; any other details of its structure are irrelevant.

Being a Casimir operator means that we have for all $z \in L_m$ for all m

$$\left[z, C^{(2)}\right] = 0 \tag{4.4}$$

i.e.

$$\sum_{k=0}^{p} \left(Q_{m+k,-k} + R_{m-k,k}\right) = 0 \tag{4.5}$$

where

$$Q_{m+k,-k} = \sum_i \left(\left[z, x^i\right] y^i + y^i \left[z, x^i\right]\right) \tag{4.6}$$

$$R_{m-k,k} = \sum_i \left(\left[z, y^i\right] x^i + x^i \left[z, y^i\right]\right) . \tag{4.7}$$

Now consider a particular term in (4.5), say $Q_{\alpha\beta}$. It is a sum of symmetrized products of a generator from L_α and a generator from L_β. Therefore it cannot possibly cancel against any other term in (4.5) except for the terms $Q_{\beta\alpha}$, $R_{\alpha\beta}$ and $R_{\beta\alpha}$, assuming that such terms are present for the particular values of α and β. We find that (4.5) breaks up into the following equations:

$$\begin{gathered} Q_{\alpha\beta} + Q_{\beta\alpha} = 0 \qquad R_{ab} + R_{ba} = 0 \qquad R_{\alpha b} + Q_{b\alpha} = 0 \\ R_{aB} + R_{Ba} + Q_{aB} = 0 \qquad R_{\alpha B} + Q_{\alpha B} + Q_{B\alpha} = 0 \\ R_{AB} + R_{BA} + Q_{AB} + Q_{BA} = 0 \end{gathered} \tag{4.8}$$

where for $N = 2p$ we have

$$0 < a, b < p \qquad p < \alpha, \beta < 2p \qquad A, B = 0 \text{ or } p \tag{4.9}$$

while for $N = 2p + 1$ we have

$$0 < a, b \leqslant p \qquad p < \alpha, \beta \leqslant 2p \qquad A, B = 0 . \tag{4.10}$$

The sum of the subscripts in (4.8) is always equal to m and the number of equations in (4.8) for fixed m is equal to $p + 1$, except that for $N =$ even and $m =$ odd it is equal to p.

Next we form the contraction L^ϵ of the Lie algebra L by applying (2.5). We define the contraction $C^{(2)\epsilon}$ of the quadratic Casimir operator $C^{(2)}$ by

$$C^{(2)\epsilon} = \sum_{k=0}^{p} \mu_k P_{k,-k} \tag{4.11}$$

where the parameters μ_k are to be determined such that for all $z \in L_m$ for all m we have

$$\left[z, C^{(2)\epsilon}\right]_\epsilon = 0 \tag{4.12}$$

i.e.

$$\sum_{k=0}^{p} \mu_k \left(\epsilon_{mk} Q_{m+k,-k} + \epsilon_{m,-k} R_{m-k,k}\right) = 0 . \tag{4.13}$$

We now make the crucial observation that the quantities $P_{k,-k}$, $Q_{m+k,-k}$ and $R_{m-k,k}$ are given by *precisely the same expressions as for the uncontracted algebra, i.e. that the effect of the* contraction is completely taken care of by the explicitly appearing μ_k, ϵ_{mk} and $\epsilon_{m,-k}$. To see this, consider, for example, $Q_{m+k,-k}$, given before contraction by (4.6):

$$Q_{m+k,-k} = \sum_i \left(u^i y^i + y^i u^i\right) \qquad u^i \in L_{m+k} \qquad y^i \in L_{-k} \tag{4.14}$$

where we have written for convenience u^i for $\left[z, x^i\right]$.

The only way the piece $\sum_i u^i y^i$ can be affected by the contraction is if it is of the form

$$\sum_i u^i y^i = vw - wv + \text{other terms} \tag{4.15}$$

since the commutator $[v, w]$ could be modified by the contraction. However, then necessarily

$$\sum_i y^i u^i = wv - vw + \text{other terms} \tag{4.16}$$

i.e. the commutator $[v, w]$ drops out upon the symmetrization inherent in the definition of $Q_{m+k,-k}$.

Next we observe that, by the same argument as before, (4.13) actually breaks up into the following equations:

$$\begin{aligned}
&\mu_{-\beta}\epsilon_{\alpha+\beta,-\beta} Q_{\alpha\beta} + \mu_{-\alpha}\epsilon_{\alpha+\beta,-\alpha} Q_{\beta\alpha} = 0 \\
&\mu_b\epsilon_{a+b,-b} R_{ab} + \mu_a\epsilon_{a+b,-a} R_{ba} = 0 \\
&\mu_b\epsilon_{\alpha+b,-b} R_{\alpha b} + \mu_{-\alpha}\epsilon_{\alpha+b,-\alpha} Q_{b\alpha} = 0 \\
&\mu_B\epsilon_{a+B,B}\left(R_{aB} + Q_{aB}\right) + \mu_a\epsilon_{a+B,-a} R_{Ba} = 0 \\
&\mu_B\epsilon_{\alpha+B,B}\left(R_{\alpha B} + Q_{\alpha B}\right) + \mu_{-\alpha}\epsilon_{\alpha+B,-\alpha} Q_{B\alpha} = 0 \\
&\mu_B\epsilon_{A+B,B}\left(R_{AB} + Q_{AB}\right) + \mu_A\epsilon_{A+B,A}\left(R_{BA} + Q_{BA}\right) = 0
\end{aligned} \tag{4.17}$$

and we obtain explicit relations between the μs and the ϵs by comparison with (4.8). (Note that in the last three equations of (4.17) we exploited the fact that $A = -A$, $B = -B$.) Thus, we find

$$\begin{aligned}
&\mu_{-\beta}\epsilon_{\alpha+\beta,-\beta} = \mu_{-\alpha}\epsilon_{\alpha+\beta,-\alpha} \qquad && \mu_b\epsilon_{a+b,-b} = \mu_a\epsilon_{a+b,-a} \\
&\mu_b\epsilon_{\alpha+b,-b} = \mu_{-\alpha}\epsilon_{\alpha+\beta,-\alpha} && \mu_B\epsilon_{a+B,B} = \mu_a\epsilon_{a+B,-a} \\
&\mu_B\epsilon_{\alpha+B,B} = \mu_{-\alpha}\epsilon_{\alpha+B,-\alpha} && \mu_A\epsilon_{A+B,A} = \mu_B\epsilon_{A+B,B}\,.
\end{aligned} \tag{4.18}$$

Note that the first of these equations is empty for $\alpha = \beta$, the second is empty for $a = b$ and the last is empty for $A = B$. These equations are valid in the so-called generic case when

$$0 \neq \left[L_j, L_k\right] \qquad \text{for all } j \text{ and } k. \tag{4.19}$$

In the special case when for some particular values of j and k the above commutator vanishes identically (i.e. for every choice of elements from the grading subspaces) the terms containing ϵ_{jk} should be *omitted* from (4.18).

Quite explicitly suppose that the grading group is Z_2. Then $p = 1$, the range of values of a, b, α, β is empty and we obtain in the generic case from the last equation of (4.18) for $A = 0,\ B = 1$ the single constraint

$$\mu_0\epsilon_{10} = \mu_1\epsilon_{11}\,. \tag{4.20}$$

The solutions of (2.5) for the possible contractions in the case of Z_2 are as follows [2, 3]. there are two trivial solutions,

$$\epsilon = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.21}$$

which determine no contraction $L = L^\epsilon$, and Abelian L^ϵ respectively, and three non-trivial ones

$$\epsilon = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.22}$$

where we use matrix notation $\epsilon = \left(\epsilon_{jk}\right)$. Correspondingly, we obtain from (4.20)

$$\begin{array}{ll} \epsilon & \mu \\ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & \mu_0 = 0,\ \mu_1 \text{ arbitrary} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \mu_0 \text{ arbitrary, } \mu_1 = 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \mu_0 \text{ arbitrary, } \mu_1 \text{ arbitrary}\,. \end{array} \tag{4.23}$$

As a second example we take for the grading group Z_3. Now again $p = 1$ and the values of the grading labels are 0, 1 and 2, i.e. we have one A-type (0), one a-type (1) and one α-type (2) label. Hence we obtain in the generic case from (4.18) the three constraints

$$\mu_1\epsilon_{02} = \mu_1\epsilon_{01} \qquad \mu_0\epsilon_{10} = \mu_1\epsilon_{12} \qquad \mu_0\epsilon_{20} = \mu_1\epsilon_{21}\,. \tag{4.24}$$

Now there are, for the generic case, 13 non-trivial solutions of (2.5) which are listed in [2] and labelled from I to XIII. Using (4.24) we find

$$\begin{array}{ll} \epsilon & \mu \\ \text{I, II, III} & \mu_0 = 0,\ \mu_1 \text{ arbitrary} \\ \text{IV, V, VI, XII, XIII} & \mu_0 \text{ arbitrary, } \mu_1 \text{ arbitrary} \\ \text{VII, VIII} & \mu_0 = 0,\ \mu_1 = 0 \\ \text{IX, X, XI} & \mu_0 \text{ arbitrary, } \mu_1 = 0 \end{array} \tag{4.25}$$

which illustrates the variety of possibilities that can result from contraction.

5. Contraction of Casimir operators of higher degree

We explain the procedure for treating Casimir operators of higher degree on the example of the cubic Casimir $C^{(3)}$, which should make clear the generalization to any degree. Let

$$C^{(3)} \in U_j \tag{5.1}$$

and let

$$C^{(3)} = \sum_{k+\ell+m=j} P_{k\ell m} \tag{5.2}$$

where

$$P_{k\ell m} \equiv \sum_i \left(x^i y^i z^i\right)_{\rm sym} \equiv \sum_i \left(x^i y^i z^i + y^i z^i x^i + z^i x^i y^i + y^i x^i z^i + x^i z^i y^i + z^i y^i x^i\right)$$
$$x^i \in L_k \qquad y^i \in L_\ell \qquad z^i \in L_m\,. \tag{5.3}$$

The requirement that $C^{(3)}$ be a Casimir means that for all $v \in L_r$ for all r we have

$$\left[v,\ C^{(3)}\right] = 0 \tag{5.4}$$

i.e.

$$\sum_{k+\ell+m=j} \left(Q_{r+k,\ell,m} + R_{k,r+\ell,m} + S_{k,\ell,r+m}\right) = 0 \tag{5.5}$$

where

$$Q_{r+k,\ell,m} \equiv \sum_i \left(\left[v, x^i\right] y^i z^i\right)_{\rm sym} \tag{5.6}$$

$$R_{k,r+\ell,m} \equiv \sum_i \left(\left[v, y^i\right] z^i x^i\right)_{\rm sym} \tag{5.7}$$

$$S_{k,\ell,r+m} \equiv \sum_i \left(\left[v, z^i\right] x^i y^i\right)_{\rm sym}\,. \tag{5.8}$$

Equation (5.5) is the analogue of (4.5) for the quadratic Casimir. The sum in (5.5) can be rearranged into

$$\sum_{s+t+p=j+r} \sum_\sigma \left(Q_{stp} + R_{stp} + S_{stp}\right) = 0 \tag{5.9}$$

where $\sum_\sigma$ means a sum over permutations of s, t and p. For the same reasons as discussed in connection with (4.5) we have now that (5.9) breaks up into the system of equations

$$\sum_\sigma \left(Q_{stp} + R_{stp} + S_{stp}\right) = 0 \tag{5.10}$$

where we have one such equation for every allowed choice of s, t and p. Equation (5.10) is the analogue of (4.8). Just as in that case, for any given choice of s, t and p not every Q, R or S need be present nor every permutation of s, t and p need occur.

We next define the contraction $C^{(3)\epsilon}$ of the cubic Casimir $C^{(3)}$ by

$$C^{(3)\epsilon} = \sum_{k+\ell+m=j} \mu_{k\ell m} P_{k\ell m} \tag{5.11}$$

with the $\mu_{k\ell m}$ given by the requirement that for every $v \in L_r$ for all r we must have

$$\left[v, C^{(3)\epsilon}\right]_\epsilon = 0 \tag{5.12}$$

i.e.

$$\sum_{k+\ell+m=j} \mu_{k\ell m} \left(\epsilon_{rk} Q_{r+k,\ell,m} + \epsilon_{r\ell} R_{k,r+\ell,m} + \epsilon_{rm} S_{k,\ell,r+m}\right) = 0 \tag{5.13}$$

which is the analogue of (4.13). Finally, by using the relations (5.10) in (5.13) we obtain a system of equations in the Casimir contraction parameters $\mu_{k\ell m}$ and the Lie algebra contraction parameters ϵ_{rk}, $\epsilon_{r\ell}$, and ϵ_{rm} only—the analogue of (4.18).

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